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# GRUSIN OPERATOR AND HEAT KERNEL OF NILPOTENT LIE GROUPS(Developments of Cartan Geometry and Related Mathematical Problems)

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## GRUSIN OPERATOR AND HEAT KERNEL ON NILPOTENT LIE GROUPS

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**ABSTRACT.** The purpose of this note is to overview how we can construct the heat kernel for (sub)-Laplacian in an explicit (integral) form with special functions. Of course such cases will be highly limited. Nevertheless there will be lots of operators interesting on nilpotent Lie groups. We will concentrate for the operators on nilpotent Lie groups and their quotient spaces. Here we only treat with typical low-dimensional cases. So first we discuss the heat kernel for Grusin operator in relation with the Mehler formula and Hamilton-Jacobi theory and explain a general integral form of heat kernel on nilpotent Lie groups from this point of view. And then we state a relation between the heat kernel on Heisenberg group and that for Grusin operator. Also we construct an classical action integral for a higher step Grusin operator.

### CONTENTS

|   |    |
|---|----|
| 1. Introduction                         | 1  |
| 2. Grusin operator                      | 3  |
| 3. Heat kernel on nilpotent groups      | 5  |
| 4. Heisenberg group and Grusin operator | 6  |
| 5. Higher step Grusin operator          | 7  |
| References                              | 11 |

### 1. INTRODUCTION

It is known in the statistical mechanics that the heat kernel  $K_t(x, y)$  is expressed as a path integral

$$K(t, x, y) = \int_{P_t(x, y)} e^{-S_t(\gamma)} d\mu(\gamma),$$

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hopefully with a suitable “infinite dimensional measure”  $d\mu(\gamma)$ , where  $P_t(x, y)$  denote the path space connecting  $x$  to  $y$  at a time  $t$  and the function  $S_t(\gamma)$  is called the classical action and is given by

$$S_t(\gamma) = 1/2 \int_0^t \|\dot{\gamma}(s)\|^2 ds.$$

By normalizing the time parameter  $t = 1$ , this is also written as

$$\frac{1}{t^N} \int_{P_1(x, y)} e^{-\frac{S_1(\gamma_t)}{2t}} d\mu(\gamma), \quad (\gamma_t(\sigma) = \gamma(\sigma t))$$

and in the Laplacian case it has an asymptotic expansion

$$K(t, x, y) \sim \frac{1}{(2\pi t)^{n/2}} e^{-\frac{d(x, y)^2}{2t}} u_0(x, y) (1 + O(t)).$$

Here  $d(x, y)$  denotes the Riemannian distance of the point  $x$  and  $y$ . For the sub-Laplacian cases the small time asymptotic expansion is more complicated (cf. [2]). There are interesting arguments which will give us a reduction of the physics formula to a mathematically fixed formula in a certain case (cf. [13]). Especially, if the space we are working on is Euclidean, then we have only one segment (geodesics) which connects  $x$  and  $y$ , and the path integral will reduce to just a function

$$\frac{1}{(2\pi t)^{n/2}} e^{-\frac{\|x-y\|^2}{2t}},$$

the heat kernel of the Laplacian  $\Delta = -\frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ .

It was the first in the paper [12] that by a probabilistic argument the heat kernel of the sub-Laplacian on three dimensional Heisenberg group was given in an explicit integral formula, and then many papers were published to express the heat kernel for Laplacians and sub-Laplacians on nilpotent Lie groups (see [1], [2], [3], [15] and also recent papers [13], [5] or [6] for dealing with similar subject and calculations). In case of the sub-Laplacian we will necessarily have an integral expression for the heat kernel, even if it reduces to a fixed finite dimensional integral expression since there are many geodesics connecting two points even locally. The first step of this is how we can suppose that the formula looks like?

So in §2 we explain the case of Grusin operator  $\mathcal{G} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2} \right)$  following a standard way of the spectral decomposition of selfadjoint operators and arrive at a form as a natural conclusion. Then we discuss the functions in the formula from Hamilton-Jacobi theory. In §3 we state a possible formula for the heat kernel on general nilpotent Lie groups and explain an action integral and transport equation which will be satisfied by the functions appearing in the formula. In §4 we give a relation between heat kernels on Heisenberg group and Grusin operator in terms of fiber integration. Finally in §5 we solve a Hamilton system for a higher step Grusin operator and construct an action integral based on the existence of the solution of the Hamilton system.

## 2. GRUSIN OPERATOR

Let  $\mathcal{G} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2} \right)$  be Grusin operator and denote by

$$\mathcal{F} : L_2(\mathbb{R}^2, dx dy) \longrightarrow L_2(\mathbb{R}^2, dx d\eta)$$

$$\mathcal{F}(\varphi)(x, \eta) = \int_{\mathbb{R}^2} e^{-\sqrt{-1}y\eta} \varphi(x, y) dy$$

a partial Fourier transformation.

Through this partial Fourier transformation, Grusin operator  $\mathcal{G}$  is seen as

$$\mathcal{L} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - x^2 \eta^2 \right),$$

acting on  $L_2(\mathbb{R}^2, dx d\eta)$ . When we regard the variable  $\eta$  as a constant, the heat kernel  $\mathcal{L}_\eta(t, x, \hat{x})$  of the operator

$$L_\eta = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - x^2 \eta^2 \right)$$

for  $\eta \neq 0$  is expressed as the sum of eigenfunctions  $V_n(x)$

$$(L_\eta V_n)(x) = \frac{(2n+1)|\eta|}{2} V_n(x),$$

where

$$V_n(x) = e^{-1/2|\eta|x^2} H_n(\sqrt{|\eta|x})$$

and

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}$$

is the  $n$ -th Hermite polynomial:

$$\begin{aligned} \mathcal{K}_\eta(t, x, \hat{x}) &= \sum_{n=1}^{\infty} e^{-\frac{2n+1}{2}|\eta|t} \frac{\sqrt{|\eta|} V_n(x) V_n(\hat{x})}{\sqrt{\pi} 2^n n!} \\ (2.1) \quad &= \sqrt{|\eta|} e^{-\frac{1}{2}|\eta|t} e^{-\frac{|\eta|}{2}(x^2 + \hat{x}^2)} \sum \frac{H_n(\sqrt{|\eta|x}) H_n(\sqrt{|\eta|\hat{x}})}{\sqrt{\pi} 2^n n!} e^{-nt|\eta|}. \end{aligned}$$

Then by the Mehler formula (cf. [17]) we know that (2.1) equals to

$$(2.2) \quad \frac{1}{\sqrt{\pi}} \sqrt{\frac{\eta}{e^{t\eta} - e^{-t\eta}}} e^{-\frac{\eta}{4} \left\{ (x+\hat{x})^2 \tanh \frac{\eta t}{2} + (x-\hat{x})^2 \coth \frac{\eta t}{2} \right\}}.$$

Note that  $|\eta| \tanh |\eta| = \eta \tanh \eta$  (and so on).

Also we have

$$\begin{aligned} &\lim_{\eta \rightarrow 0} \mathcal{K}_\eta(t, x, \hat{x}) \\ &= \lim_{\eta \rightarrow 0} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\eta}{\sinh t\eta}} e^{-\frac{\eta}{4} \left\{ (x+\hat{x})^2 \tanh \frac{\eta t}{2} + (x-\hat{x})^2 \coth \frac{\eta t}{2} \right\}} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-\hat{x}|^2}{2t}} \end{aligned}$$

is the heat kernel for the operator  $-\frac{1}{2} \frac{d^2}{dx^2}$ .

Now we have the heat kernel  $K^\mathcal{G} = K^\mathcal{G}(t, x, y, \hat{x}, \hat{y})$  of the Grusin operator  $\mathcal{G} = \mathcal{F}^{-1} \circ \mathcal{L} \circ \mathcal{F}$ ;

$$\begin{aligned}
(e^{-t\mathcal{G}}f)(x, y) &= \int K^{\mathcal{G}}(t, x, y, \hat{x}, \hat{y}) f(\hat{x}, \hat{y}) d\hat{x}d\hat{y}; \\
K^{\mathcal{G}}(t, x, y, \hat{x}, \hat{y}) \\
(2.3) \quad &= \frac{1}{(2\pi)^{3/2}} \int e^{\sqrt{-1}(y-\hat{y})\eta} e^{-\frac{\eta}{4}\{(x+\hat{x})^2 \tanh \frac{t\eta}{2} + (x-\hat{x})^2 \coth \frac{t\eta}{2}\}} \sqrt{\frac{\eta}{\sinh t\eta}} d\eta.
\end{aligned}$$

In the above expression, if we still change the variable  $t\eta$  to  $\eta$  (= time rescaling), then it becomes the following form :

$$\begin{aligned}
K(t, x, y, \hat{x}, \hat{y}) \\
(2.4) \quad &= \frac{1}{(2\pi t)^{3/2}} \int e^{\frac{\sqrt{-1}(y-\hat{y})\eta}{t}} e^{-\frac{\eta}{4t}\{(x+\hat{x})^2 \tanh \frac{\eta}{2} + (x-\hat{x})^2 \coth \frac{\eta}{2}\}} \sqrt{\frac{\eta}{\sinh \eta}} d\eta.
\end{aligned}$$

Put

$$S(x, \hat{x}, \eta) = \frac{\eta}{4} \left\{ (x + \hat{x})^2 \tanh \frac{\eta}{2} + (x - \hat{x})^2 \coth \frac{\eta}{2} \right\}$$

and  $V(\eta) = \sqrt{\frac{\eta}{\sinh \eta}}$ , and then we write this integral form (2.4) as

$$(2.5) \quad \frac{1}{(2\pi t)^{3/2}} \int e^{\frac{\sqrt{-1}(y-\hat{y})\eta}{t}} e^{-\frac{S(x, \hat{x}; \eta)}{t}} V(\eta) d\eta.$$

Now we construct the function  $S = S(x, \hat{x}; \eta)$  by solving a Hamilton system with the Hamiltonian  $2H^\eta = 2H^\eta(x, \xi) = \xi^2 - x^2\eta^2$ :

$$(2.6) \quad \begin{cases} \dot{x}(s) = \frac{\partial H^\eta}{\partial \xi} = \xi, \\ \dot{\xi}(s) = -\frac{\partial H^\eta}{\partial x} = x\eta^2, \\ \text{boundary condition : } x(0) = x, \quad x(t) = \hat{x}. \end{cases}$$

In fact this system is solved explicitly with the solution that

$$\begin{aligned}
x(s) &= x(s; t, x, \hat{x}, \eta) = \frac{\hat{x} \sinh s\eta + x \sinh \eta(t-s)}{\sinh t\eta} \\
\xi(s) &= \xi(s; t, x, \hat{x}, \eta) = \dot{x}(s) = \eta \frac{\hat{x} \cosh s\eta - x \cosh(t-s)\eta}{\sinh t\eta},
\end{aligned}$$

for any  $t, x, \hat{x} \in \mathbb{R}$ . With this solution, let  $\varphi = \varphi(x, \hat{x}, t; \eta)$  be the integral

$$(2.7) \quad \varphi(x, \hat{x}, t; \eta) = \int_0^t \dot{x}(s)\xi(s) - H^\eta(x(s), \xi(s)) ds.$$

This integral is called a classical action and is equal to

$$\begin{aligned}
\varphi(x, \hat{x}, t; \eta) &= \eta^2 \int_0^t x^2(s) ds + \frac{t}{2} E \\
(2.8) \quad &= \frac{\eta}{4} \left\{ (\hat{x} + x)^2 \tanh \frac{t\eta}{2} + (\hat{x} - x)^2 \coth \frac{t\eta}{2} \right\},
\end{aligned}$$

where the constant

$$\begin{aligned} E &\equiv \xi^2(s) - x^2(s)\eta^2 = \xi^2(0) - x^2\eta^2 \\ &= \frac{\eta^2 \{4(x^2 + \hat{x}^2) - 4x\hat{x}(e^{t\eta} + e^{-t\eta})\}}{(e^{t\eta} - e^{-t\eta})^2} \end{aligned}$$

is an invariant of the Hamilton system (2.6).

Here we know that the function  $\varphi(x, \hat{x}, 1; \eta)$  coincides with the function  $S$  and it is a solution of the Hamilton-Jacobi equation:

$$(2.9) \quad \frac{\partial}{\partial t} \varphi(x, \hat{x}, t; \eta) + H^\eta\left(\hat{x}, \frac{\partial}{\partial \hat{x}} \varphi(x, \hat{x}, t; \eta)\right) = 0.$$

The function  $\varphi$  has a property that  $\varphi(x, \hat{x}, 1; t\eta) = t\varphi(x, \hat{x}, t; \eta)$ , and this implies that the function  $S$  satisfies the equation, called generalized Hamilton-Jacobi equation:

$$(2.10) \quad H^\eta\left(\hat{x}, \frac{\partial}{\partial \hat{x}} S(x, \hat{x}, \eta)\right) + \eta \frac{\partial}{\partial \eta} S(x, \hat{x}, \eta) = S(x, \hat{x}, \eta).$$

For each fixed  $t$ ,  $x$  and  $\eta$  let  $\mathcal{V} : \hat{x} \mapsto \xi(0; t, x, \hat{x}, \eta)$ , then by the explicit expression of the solution of the Hamilton system we have

$$(2.11) \quad \mathcal{V}(\hat{x}) = \frac{\eta}{\sinh t\eta} (\hat{x} - x \cosh t\eta),$$

and

$$(2.12) \quad \sqrt{\frac{\partial \mathcal{V}}{\partial \hat{x}}}(\hat{x}) = \sqrt{\frac{\eta}{\sinh t\eta}}.$$

This function  $\sqrt{\frac{\partial \mathcal{V}}{\partial \hat{x}}}$ , as a function of the parameter  $\eta$  (with  $t = 1$ ), is a solution of the transport equation (3.6)(see §3).

Summing up, we know that the functions in the integral form (2.3) coincide with the functions (2.8) and (2.12) we constructed by solving the Hamilton system (2.6). In fact by putting  $t = 1$  (time rescaling) we have the functions  $S = S(x, \hat{x}, \eta) = \varphi(x_0, x, 1; \eta)$  and  $\sqrt{\frac{\partial \mathcal{V}}{\partial \hat{x}}}(\hat{x}) = V(\eta)$ (cf. [18], [10]).

### 3. HEAT KERNEL ON NILPOTENT GROUPS

On the Lie group  $G$  the heat kernel for the (left)invariant (sub)-Laplacian  $\Delta = -\frac{1}{2} \sum \tilde{X}_i^2$  takes the form  $k(t, g^{-1} \cdot h)$  with a smooth function  $k(t, g) \in C^\infty(\mathbb{R}_+ \times G)$  satisfying

$$(3.1) \quad \left(\frac{\partial}{\partial t} + \Delta\right)k(t, g) = 0$$

$$(3.2) \quad \lim_{t \downarrow 0} k(t, g) = \delta_e, \quad \delta_e \text{ is the } \delta \text{ function at the identity element } e \in G.$$

So, if the heat kernel would be given by a function  $k(t, g)$  of an integral form

$$(3.3) \quad \frac{1}{t^N} \int e^{-\frac{t(g, \eta)}{t}} V(g, \eta) d\eta$$

with a function  $f = f(g, \eta) \in C^\infty(G \times \mathbb{R}^\ell)$  which we take a function defined by the integral similar to (2.7) with a modification of the term  $\sqrt{-1}(y - \hat{y})\eta$  (we call  $f$  a *complex action function*), then the function  $V = V(g, \eta)$  (we call it a *volume element*) will satisfy an equation (called *transport equation*).

Now we shall state these equations. Let  $H$  be the Hamiltonian of the (sub)-Laplacian  $\Delta$  and the function  $f$  satisfies the equation, called **generalized Hamilton-Jacobi equation**:

$$(3.4) \quad H(x, \nabla f) + \sum \eta_i \frac{\partial}{\partial \eta_i} f(x, \eta) = f(x, \eta).$$

And then with one solution of this equation we assume that the function  $V$  satisfy the equation, called **transport equation**:

$$(3.5) \quad \sum \eta_i \frac{\partial V}{\partial \eta_i} + \left( \sum_i \tilde{X}_i(f) \tilde{X}_i(V) - (\Delta(f) + N - \ell) \cdot V \right) = 0.$$

Especially, if the function  $V$  does not depend on the space variables then, this equation reduces to

$$(3.6) \quad \sum \eta_i \frac{\partial V}{\partial \eta_i} - (\Delta(f) + N - \ell) V = 0.$$

The function (2.12) is a solution of this equation.

If we have these two functions  $f$  and  $V$  satisfying the generalized Hamilton-Jacobi equation and transport equation then these will give the heat kernel. In the paper [1] it was proved that for the sub-Laplacian on any two step nilpotent Lie group the heat kernel is given by the integral (3.3) with a complex action  $f$  and a volume element  $V$ . In fact both of them are explicitly given in terms of hyperbolic functions. The complex action function is constructed by solving a Hamilton system ( $\sim$  bi-characteristic equation) under initial-boundary conditions and the volume element is constructed from the Jacobian of the correspondence similar to the map (2.11) between boundary condition and initial condition of the Hamilton system (see [8] for an aspect of functional calculus).

#### 4. HEISENBERG GROUP AND GRUSIN OPERATOR

In this section we just describe the heat kernel of the three dimensional Heisenberg group and discuss a relation with that for Grusin operator.

Let  $\mathbf{H}$  be the three dimensional Heisenberg group and denote its Lie algebra by  $\mathfrak{h}$  whose basis we denote by  $\{X, Y, Z\}$  with the bracket relation  $[X, Y] = Z$ . We identify  $\mathbf{H}$  and  $\mathfrak{h}$  through the exponential map  $\exp : \mathfrak{h} \rightarrow \mathbf{H}$  and denote by  $\tilde{X}$ ,  $\tilde{Y}$  and  $\tilde{Z}$  the left invariant vector fields corresponding to  $X$ ,  $Y$  and  $Z$  respectively. Then  $\Delta_{sub} = -\frac{1}{2}(\tilde{X}^2 + \tilde{Y}^2)$  is a sub-Laplacian on  $\mathbf{H}$ . Let  $\mathbf{N}_Y = [\{tY\}_{t \in \mathbb{R}}]$  be a subgroup generated by the element  $Y$ . The map  $\rho : \mathbf{H} \rightarrow \mathbb{R}^2$  defined by

$$\rho : \mathbf{H} \cong \mathfrak{h} \ni g = xX + yY + zZ = (x, y, z) \mapsto (u, v) \in \mathbb{R}^2$$

$$u = x, \quad v = z + \frac{1}{2}xy$$

realizes the projection map

$$\mathbf{H} \cong \mathbb{R}^3 \rightarrow \mathbf{N}_Y \backslash \mathbf{H} \cong \mathbb{R}^2.$$

In fact, this is a principal bundle and the trivialization is given by the map

$$(4.1) \quad \begin{aligned} \mathbf{N}_Y \times (\mathbf{N}_Y \backslash \mathbf{H}) &\cong \mathbb{R} \times \mathbb{R}^2 \ni (a; u, v) \mapsto (x, y, z) \in \mathbb{R}^3 \cong \mathbf{H} \\ (a; u, v) &\mapsto (u, a, v - \frac{1}{2}au). \end{aligned}$$

Then the left invariant vector field  $\tilde{X} = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}$  descends to the vector field  $\frac{\partial}{\partial u}$  and  $\tilde{Y} = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}$  descends to  $u \frac{\partial}{\partial v}$ . So the sub-Laplacian  $\Delta_{sub}$  on  $\mathbf{H}$  and Grusin operator commutes each other through the map  $\rho$ :

$$(4.2) \quad \Delta_{sub} \circ \rho^* = \rho^* \circ \mathcal{G}.$$

By the left invariance of  $\Delta_{sub}$ , the heat kernel  $K^{\mathbf{H}}(t; g, h) \in C^\infty(\mathbb{R}_+ \times \mathbf{H} \times \mathbf{H})$  of  $\Delta_{sub}$  takes the form  $K^{\mathbf{H}}(t; g, h) = k^{\mathbf{H}}(t; g^{-1} \cdot h)$  with a smooth function  $k^{\mathbf{H}}(t, g) \in C^\infty(\mathbb{R}_+ \times \mathbf{H})$ . This function is given as (cf. [1]):

$$(4.3) \quad k^{\mathbf{H}}(t, g) = k^{\mathbf{H}}(t, x, y, z) = \frac{1}{(2\pi t)^2} \int e^{-\frac{\sqrt{-1}\eta x \cdot \frac{y}{2} \coth \frac{\eta}{2} \cdot (x^2 + y^2)}{t}} \frac{\eta}{2 \sinh \frac{\eta}{2}} d\eta$$

Now by (4.1) and (4.2) we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} K^{\mathbf{H}}(t, (x, y, z), (u, a, v - 1/2ua)) da \\ &= K^{\mathcal{G}}(t, (x, z + 1/2xy), (u, v)) \end{aligned}$$

that is, the fiber integration of the function  $K^{\mathbf{H}}(t; g, h)$  along the fiber of the map  $\rho$  gives the heat kernel of the Grusin operator.

## 5. HIGHER STEP GRUSIN OPERATOR

Higher step Grusin operator is defined as

$$(5.1) \quad \mathcal{G}^{(k)} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + x^{2k} \frac{\partial^2}{\partial y^2} \right).$$

All these comes from a sub-Laplacian on a suitable nilpotent Lie group  $\mathbf{G}_{k+1}$ , i.e., let  $\mathfrak{g}_{k+1}$  be a Lie algebra with the basis  $\{X_0, \dots, X_k\}$  such that bracket relations are defined by

$$[X_0, X_1] = X_2, [X_0, X_2] = X_3, \dots, [X_0, X_{k-1}] = X_k, [X_0, X_k] = 0,$$

and all other are zero.  $\mathbf{G}_{k+1}$  is the corresponding simply connected group and we identify it with  $\mathfrak{g}_{k+1}$  through the exponential map.

Let  $\mathcal{N}$  be a subgroup of  $\mathbf{G}_{k+1}$  generated by  $\{X_1, \dots, X_{k-1}\}$ , then  $\mathcal{N} \backslash \mathbf{G}_{k+1}$  is isomorphic to  $\mathbb{R}^2$  and the sub-Laplacian on  $\mathbf{G}_{k+1}$

$$-\frac{1}{2} (\tilde{X}_0^2 + \tilde{X}_1^2)$$

descends to

$$-\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + x^{2k} \frac{\partial^2}{\partial y^2} \right).$$



This group is a special class of Carnot group (cf. [5]), and Engel group is such an one of dimension 4 (we ignore a constant in front of  $x^{2k}$ ).

Until now we have no explicit expression for the heat kernel on nilpotent Lie group of step greater than 3. In this final section we construct the classical action integral of a higher step Grusin operator

$$\mathcal{G}^{(2)} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + x^4 \frac{\partial^2}{\partial y^2} \right).$$

Through the partial Fourier transformation  $\mathcal{F}$  we consider the operator (cf. [19], [20])

$$L_\eta^{(2)} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - x^4 \eta^2 \right).$$

As in §1, for each fixed  $\eta \neq 0$  the heat kernel  $\mathcal{K}^{L_\eta^{(2)}}(t, x, \widehat{x})$  of the operator  $L_\eta^{(2)}$  has a form

$$\mathcal{K}^{L_\eta^{(2)}}(t, x, \widehat{x}) = \sqrt[3]{|\eta|} \cdot \sum e^{-t(\sqrt[3]{|\eta|})^2 \lambda_k} \varphi_k \left( \sqrt[3]{|\eta|} x \right) \varphi_k \left( \sqrt[3]{|\eta|} \widehat{x} \right)$$

with the normalized eigenfunctions of  $L_1^{(2)}$ :

$$\left( L_1^{(2)} \varphi_k \right) (x) = \lambda_k \varphi_k(x), \quad 0 \leq \lambda_1 \leq \lambda_2, \dots, \quad \int |\varphi(x)|^2 dx = 1.$$

The heat kernel of a higher step Grusin operator  $\mathcal{F}^{-1} \circ \mathcal{G}_2 \circ \mathcal{F} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + x^4 \frac{\partial^2}{\partial y^2} \right)$  will be

$$\frac{1}{2\pi} \int e^{\sqrt{-1}(y-\widehat{y})\eta} \mathcal{K}^{L_\eta^{(2)}}(t, x, \widehat{x}) d\eta.$$

*It is not clear that this has a similar form with (2.3) or (2.4). We construct here an action integral by solving a Hamilton system similar to (2.6), which solution is given in terms of elliptic functions and we will know that the action integral satisfies Hamilton-Jacobi equation.*

Let  $H^\eta = H^\eta(x, \xi) = \frac{1}{2}(\xi^2 - x^4 \eta^2)$  be the Hamiltonian of the operator  $L_\eta^{(2)}$ , and consider the Hamilton system:

$$\dot{x}(s) = \xi, \quad \dot{\xi}(s) = -H_x^\eta(x, \xi) = 2x^3 \eta^2$$

with the boundary condition

$$x(0) = x_0, \quad x(t) = x, \quad (x_0, x \text{ and } t \neq 0 \text{ should be taken arbitrary}).$$

The system reduces to a single non-linear equation:

$$(5.2) \quad \ddot{x} = 2x^3 \eta^2, \quad \text{with the boundary condition } x(0) = x_0, \quad x(t) = x.$$

It is enough to consider the cases except  $x_0 = 0 = x$ , for which we have the trivial solution  $x(s) \equiv 0$ . Then, by the transformations  $s \mapsto t - s$  and  $x(s) \mapsto -x(s)$ , it is enough to consider the two cases of the boundary data with  $t > 0$ :

$$\begin{aligned} \text{(I)} \quad & x_0 \leq 0 < x, \\ \text{(II)} \quad & 0 < x_0 \leq x. \end{aligned}$$

We describe the solution:

Case I. Let  $x_0 \leq 0 < x$ .

Let  $E > 0$  and the function  $h(y, E)$  be

$$h(y, E) = \int_{x_0}^y \frac{du}{\sqrt{u^4 \eta^2 + E}},$$

then for each fixed  $y$  the function  $h(y, E)$  is monotone as a function of  $E > 0$ , and for each fixed  $x > 0 \geq x_0$  it takes values from 0 to  $\infty$  when  $E$  moves from  $\infty$  to 0. So put  $E = E(x_0, x, t; \eta)$  be the unique constant such that

$$\int_{x_0}^x \frac{du}{\sqrt{u^4 \eta^2 + E}} = t > 0.$$

Now since the function  $h(y, E(x_0, x, t; \eta))$  ( $-\infty < y < +\infty$ ) is monotone, let  $x(s; E(x_0, x, t; \eta))$  be its inverse function, i.e.,

$$\int_{x_0}^{x(s; E(x_0, x, t; \eta))} \frac{du}{\sqrt{u^4 \eta^2 + E(x_0, x, t; \eta)}} = s,$$

then  $x(s; E(x_0, x, t; \eta))$  is the unique solution of the equation (5.2).

Case II. Let  $0 < x_0 \leq x$ . Then we need to divide into three cases.

II-1. Let  $0 < t \leq \frac{x_0^{-1} - x^{-1}}{|\eta|} = \int_{x_0}^x \frac{du}{\sqrt{u^4 \eta^2}}.$

Then for such  $t$  and  $x > x_0$  we have a unique value  $E = E(x_0, x, t; \eta) \geq 0$  such that

$$\int_{x_0}^x \frac{du}{\sqrt{u^4 \eta^2 + E}} = t.$$

The solution  $x(s; E(x_0, x, t; \eta))$  of (5.2) is then given by the integral

$$\int_{x_0}^{x(s; E(x_0, x, t; \eta))} \frac{du}{\sqrt{u^4 \eta^2 + E(x_0, x, t; \eta)}} du = s.$$

II-2.

We assume  $\frac{x_0^{-1} - x^{-1}}{|\eta|} < t \leq \int_{x_0}^x \frac{du}{\sqrt{u^4 \eta^2 - x_0^4 \eta^2}}$  and fix the unique value  $E = E(x_0, x, t; \eta)$  ( $0 > E \geq -x_0^4 \eta^2$ ) such that  $\int_{x_0}^x \frac{du}{\sqrt{u^4 \eta^2 + E(x_0, x, t; \eta)}} = t$ , then the solution of (5.2) is given by

$$\int_{x_0}^{x(s; E(x_0, x, t; \eta))} \frac{du}{\sqrt{u^4 \eta^2 + E(x_0, x, t; \eta)}} = s.$$

II-3.

Let  $t > \int_{x_0}^x \frac{du}{\sqrt{u^4 \eta^2 - x_0^4 \eta^2}}.$

Then we take the unique value  $a = a(x_0, x, t; \eta)$  ( $a(x_0, x, t; \eta)$  can be chosen uniquely in  $0 < a(x_0, x, t; \eta) < x_0$ ) such that

$$(5.3) \quad - \int_{x_0}^a \frac{du}{\sqrt{u^4 \eta^2 - a^4 \eta^2}} + \int_a^x \frac{du}{\sqrt{u^4 \eta^2 - a^4 \eta^2}} = t.$$

The monotonicity of the sum of integral (5.3) with respect to the variable  $a \in (0, x_0)$  will be seen by the coordinate change  $u = va$  in the integral.

Here put  $E = E(x_0, x, t; \eta) = -a(x_0, x, t; \eta)^4 \eta^2$ , then the unique solution of (5.2) exists and is described as follows:

Put  $s_1 = - \int_{x_0}^{a(x_0, x, t; \eta)} \frac{du}{\sqrt{u^4 \eta^2 - a(x_0, x, t; \eta)^4 \eta^2}}$ , then for  $s < s_1$  the solution  $x(s) = x(s; E(x_0, x, t; \eta))$  is defined by the integral

$$- \int_{x_0}^{x(s)} \frac{du}{\sqrt{u^4 \eta^2 + E(x_0, x, t; \eta)}} = s$$

and for  $s_1 < s$  the solution  $x(s) = x(s; E(x_0, x, t; \eta))$  is defined by the integral

$$\int_a^{x(s)} \frac{du}{\sqrt{u^4 \eta^2 + E(x_0, x, t; \eta)}} = s - s_1.$$

Note that  $\lim_{s \rightarrow s_1 \pm 0} x(s) = a(x_0, x, t; \eta)$  and  $\lim_{s \rightarrow s_1 \pm 0} \dot{x}(s) = 0$ , so this solution coincides with the solution of (5.2) under the initial condition  $x(s_1) = a(x_0, x, t)$  and  $\dot{x}(s_1) = 0$ . The case  $t \neq 0$ ,  $0 < x_0 = x$  should be understood as being included in the case **II-3**.

The solution  $x(s)$  satisfies a relation  $x(st; E(x_0, x, t; \eta)) = x(s; E(x_0, x, 1; t\eta))$ , and  $E(x_0, x, 1; t\eta) = t^2 E(x_0, x, t; \eta)$ .

Hence we could know the existence of the solution of (5.2),  $x(s; E(x_0, x, t; \eta))$ , for arbitrary boundary data  $x(0) = x_0$ ,  $x(t) = x$  ( $x_0, x, t \neq 0$  can be taken arbitrary). Although all these are expressed in terms of elliptic functions ( $sn$ -function,  $cn$ -function and so on, cf. [21] and [14]), we do not here rewrite them in terms of elliptic functions.

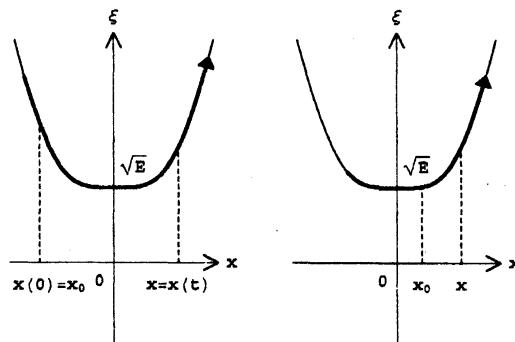


FIGURE 1.  $\xi^2 = \eta^2 x^4 + E$  with  $E > 0$  (Case I and Case II-1).

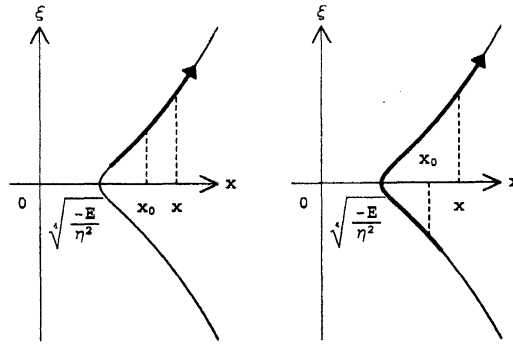


FIGURE 2.  $\xi^2 = \eta^2 x^4 + E$  with  $E < 0$  (Case II-2 and Case II-3).

Now based on the existence of the solution of (5.2) we can define the (classical) action integral  $f$ :

$$(5.4) \quad f(x_0, x, t; \eta) = \int_0^t \dot{x}(s)\xi(s) - H^\eta(x(s), \xi(s)) ds.$$

By the relation  $\dot{x}(s)^2 = \eta^2 x(s)^4 + E(x_0, x, t; \eta)$ , this integral equals to

$$\begin{aligned} f(x_0, x, t; \eta) &= \eta^2 \int_0^t x(s)^4 ds + \frac{t}{2} E(x_0, x, t; \eta) \\ &= \eta^2 \int_{x_0}^x \frac{y^4}{\sqrt{y^4 \eta^2 + E(x_0, x, t; \eta)}} dy + \frac{t}{2} E(x_0, x, t; \eta) \\ (5.5) \quad &= \pm \frac{1}{3} \left\{ x \sqrt{x^4 \eta^2 + E(x_0, x, t; \eta)} - x_0 \sqrt{x_0^4 \eta^2 + E(x_0, x, t; \eta)} \right\} + \frac{t}{6} E(x_0, x, t; \eta) \\ &\quad \left( \xi(s) = \dot{x}(s) = \pm \sqrt{x^4(s) \eta^2 + E(x_0, x, t; \eta)} \right). \end{aligned}$$

$f$  is a solution of Hamilton-Jacobi equation  $\frac{\partial}{\partial t} f + H(x, \nabla f) = 0$  and also satisfies the generalized Hamilton-Jacobi equation

$$H(x, \nabla f) + \eta \frac{\partial}{\partial \eta} f(x_0, x, 1; \eta) = f(x_0, x, 1; \eta),$$

which is proved by making use of the relation:  $t f(x_0, x, t; \eta) = f(x_0, x, 1; t\eta)$ .

Finally we note that our arguments above are also valid to show the existence of the solution for the Hamilton system (5.2) of general higher step Grusin operator and so we have an action integral similar to (5.5).

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